Cohen-Macaulay normal Rees algebras of integrally closed ideals

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Introduction

Question 1

Let $(R, \mathfrak{m})$ be a RLR with $d = \dim R$ and I an integrally closed $\mathfrak{m}$-primary ideal of R. When does the Rees algebra $\mathcal{R}(I) = \bigoplus_{n \ge 0} I^n$ become a CM normal domain?

Let R be a Noetherian ring and I an ideal of R. Recall

- $x \in R$ is integral over I $\overset{def}{\iff}$ $x^n + c_1x^{n-1} + \cdots + c_n = 0$ for $\exists\, n \ge 1$, $\exists\, c_i \in I^i$
- $I \subseteq \overline{I} = \{x \in R \mid x \text{ is integral over } I\} \subseteq R$
- I is integrally closed $\overset{def}{\iff}$ $\overline{I} = I$
- I is normal $\overset{def}{\iff}$ $\overline{I^n} = I^n$ for $\forall\, n \ge 1$.

We define

$$\mathcal{R}(I) = R[It] = \sum_{n \ge 0} I^n t^n \subseteq R[t], \quad \mathcal{R}(I) \cong \bigoplus_{n \ge 0} I^n$$

and call it the Rees algebra of I.

- The canonical morphism $f : \operatorname{Proj} \mathcal{R}(I) \to \operatorname{Spec} R$ is the blow-up of $\operatorname{Spec} R$ along the subscheme $V(I)$ defined by I.

Note that

$$\overline{\mathcal{R}(I)}^{R[t]} = \sum_{n\geq 0} \overline{I^n} t^n \cong \bigoplus_{n\geq 0} \overline{I^n} \quad \text{and} \quad \overline{\mathcal{R}(I)}^{\mathrm{Q}(\mathcal{R}(I))} = \sum_{n\geq 0} \overline{I^n \overline{R}} t^n \cong \bigoplus_{n\geq 0} \overline{I^n \overline{R}}.$$

Hence, $\mathcal{R}(I)$ is normal $\iff$ I is normal, provided R is a normal domain.

The associated graded ring of I

$$\mathrm{gr}_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1} \cong R/I \otimes_R \mathcal{R}(I)$$

plays a key role in the study of $\mathcal{R}(I)$.

Theorem 2 ([Goto-Shimoda, 1979])

Let $(R, \mathfrak{m})$ be a CM local ring with $\dim R \geq 1$ and $\sqrt{I} = \mathfrak{m}$. Then

$$\mathcal{R}(I) \text{ is CM} \iff \mathrm{gr}_I(R) \text{ is CM and } \mathrm{a}(\mathrm{gr}_I(R)) < 0.$$

- Theorem 2 holds for ideals I with $\mathrm{ht}_R\, I > 0$ ([Trung-Ikeda, 1989]).
- When R is a RLR (or more generally pseudo-rational local ring) and $I \neq R$, we have

$$\mathcal{R}(I) \text{ is CM} \iff \mathrm{gr}_I(R) \text{ is CM} \qquad \text{([Lipman, 1994])}.$$

Question 1

Let $(R, \mathfrak{m})$ be a RLR with $d = \dim R$ and I an integrally closed $\mathfrak{m}$-primary ideal of R. When does the Rees algebra $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n$ become a CM normal domain?

- Question 1 is always true when $d \leq 1$.

Preceding results

Let $(R, \mathfrak{m})$ be a RLR with $d = \dim R$ and I an integrally closed $\mathfrak{m}$-primary ideal of R.

- If $d = 2$, then $\mathcal{R}(I)$ is normal ([Zariski, 1938], [Zariski-Samuel, 1960]).
- If $d = 2$, then $\mathcal{R}(I)$ is CM ([Lipman-Teissier, 1981]).

When $d \geq 3$, we have the following examples.

Example 3

Let $R = k[[X, Y, Z]]$ be the formal power series ring over a field k. Consider

$$Q = (X^7, Y^3, Z^2) \quad \text{and} \quad I = \overline{Q} = (X^7, Y^3, Z^2, X^5Y, X^4Z, X^3Y^2, X^2YZ, Y^2Z).$$

Then $\overline{I} = I$, $\overline{I^2} \neq I^2$, and $I^2 = QI$. Hence $\mathcal{R}(I)$ is CM, but not normal.

Example 4 ([Huckaba-Huneke, 1999])

Let $R = k[[X, Y, Z]]$ be the formal power series ring over a field k. Suppose $\mathrm{ch}\, k \neq 3$. Consider

$$I = (X^4, X(Y^3 + Z^3), Y(Y^3 + Z^3), Z(Y^3 + Z^3)) + \mathfrak{m}^5$$

where $\mathfrak{m} = (X, Y, Z)$. Then I is normal and $\mathrm{gr}_I(R) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is not CM. Hence, $\mathcal{R}(I)$ is normal, but not CM.

- $v(-)$ the embedding dimension of a ring
- $\mu_R(-)$ the minimal number of generators

Preceding results

Let $(R, \mathfrak{m})$ be a RLR with $d = \dim R$ and I an integrally closed $\mathfrak{m}$-primary ideal of R.

- By [Goto, 1987], we have
  (1) $\mu_R(I) = d \implies \mathcal{R}(I)$ is a CM normal domain
  (2) $\mu_R(I) = d \iff v(R/I) \leq 1$.
- By [Ciupercă, 2006, 2011], we have
  (1) $\mu_R(I) = d + 1 \implies \mathcal{R}(I)$ is a CM normal domain
  (2) $\mu_R(I) = d + 1 \implies v(R/I) \leq 2$.

Main results

Theorem A

Let $(R, \mathfrak{m})$ be a RLR with $d = \dim R$ and $\sqrt{I} = \mathfrak{m}$ s.t. $\overline{I} = I$. Then

(1) $v(R/I) \leq 2 \implies \mathcal{R}(I)$ *is a CM normal domain*

(2) $\mu_R(I) \leq d + 2 \implies v(R/I) \leq 2$.

In particular, if $\mu_R(I) \leq d + 2$, then $\mathcal{R}(I)$ is a CM normal domain.

Why $\mu_R(I) \leq d + 2$?

Suppose $d = 3$ and $|R/\mathfrak{m}| = \infty$. Then

- $v(R/I) \leq 2 \iff I$ contains a minimal basis of $\mathfrak{m} \iff I \not\subseteq \mathfrak{m}^2$
- $\sqrt{I} = \mathfrak{m}$ and $\overline{I} = I \implies I$ is $\mathfrak{m}$-full
- $\mu_R(I) \leq d + 2 \ (= 5) \implies I \not\subseteq \mathfrak{m}^2$.

Indeed, if $I \subseteq \mathfrak{m}^2$, then

$$5 = d + 2 \geq \mu_R(I) \geq \mu_R(\mathfrak{m}^2) = \binom{d+1}{2} = \frac{d(d+1)}{2} = 6.$$

This makes a contradiction. Hence $I \not\subseteq \mathfrak{m}^2$. If $\mu_R(I) = d + 3$, then $\exists\, I = \overline{I}$ s.t. $I \subseteq \mathfrak{m}^2$.

Example 5

Let $R = k[[X, Y, Z]]$ be the formal power series ring over a field k.

- Let $I = \overline{(X^3, Y^3, Z)} = (X^3, X^2Y, XY^2, Y^3, Z)$. Then $\overline{I} = I$, $\sqrt{I} = \mathfrak{m}$, and $\mu_R(I) = 5 = d + 2$. Hence, $\mathcal{R}(I)$ is a CM normal domain.
- Let $I = \overline{(X^4, Y^4, Z)} = (X^4, X^3Y, X^2Y^2, XY^3, Y^4, Z)$. Then $\overline{I} = I$, $\sqrt{I} = \mathfrak{m}$, and $\mu_R(I) = 6 > d + 2$, but $v(R/I) = 2$. Hence, $\mathcal{R}(I)$ is a CM normal domain.
- Let $I = (f) + \mathfrak{m}^n$ for $\forall f \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $\forall n \geq 1$. Then $\overline{I} = I$, $\sqrt{I} = \mathfrak{m}$, and $v(R/I) \leq 2$. Hence, $\mathcal{R}(I)$ is a CM normal domain.
- Let $I = \overline{(X^2, Y^2, Z^4)} = (X^2, XY, Y^2, Z^4, XZ^2, YZ^2) \subseteq \mathfrak{m}^2$. Then $\overline{I} = I$, $\sqrt{I} = \mathfrak{m}$, and $v(R/I) = 3$. Since $\overline{I^2} = I^2$, the ideal I is normal. By setting $Q = (X^2, Y^2, Z^4)$, we have $I^2 = QI$. Hence, $\mathcal{R}(I)$ is a CM normal domain.

Theorem B

Let $(R, \mathfrak{m})$ be a RLR with $d = \dim R$ and $F = R^{\oplus e}$ $(e > 0)$. Let E be an R-submodule of F s.t. $\ell_R(F/E) < \infty$ and $\overline{E} = E$. Then

(1) $\mu_R([E + \mathfrak{m}F]/E) \leq 2 \implies \mathcal{R}(E)$ *is a CM normal domain*

(2) $\mu_R(E) \leq d + e + 1 \implies \mu_R([E + \mathfrak{m}F]/E) \leq 2.$

In particular, if $\mu_R(E) \leq d + e + 1$, then $\mathcal{R}(E)$ is a CM normal domain.

Thank you for your attention.